# Circular sets and powers of two

by G. Lathoud, May 2017

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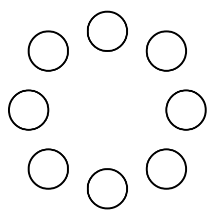
## Summary


This paper investigates a non-uniform way to visit a circular set without repetition. Interestingly, it turns out that only circular sets of $2^q$ elements permit this.

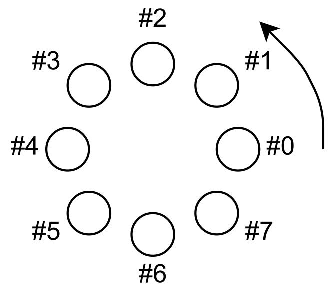

## Definition: circular set

Take $N$ holes arranged in a circle:



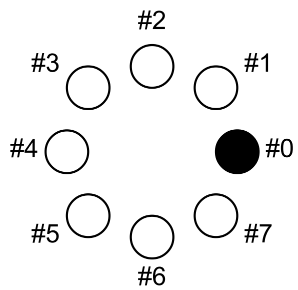
Example circular set with $N = 8$

...pick one as the first hole #0, chose a direction (e.g. counterclockwise), and name the following holes accordingly #1, #2, ..., #$(N-1)$:

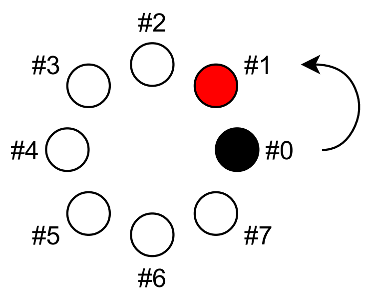



Circular set ($N = 8$) with direction and numbers

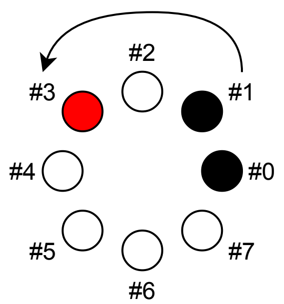
# Filling the set: an example ($N = 8$)



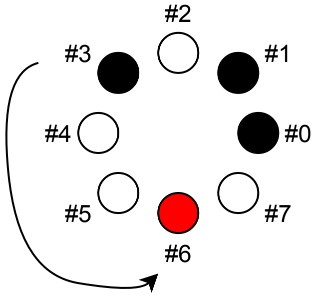

Fill the first hole #0



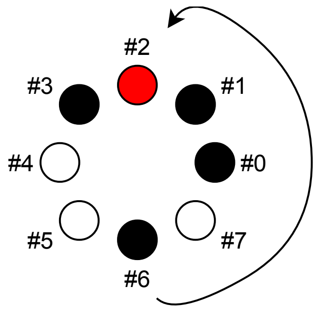

Move $i = 1$ hole forward, and fill the destination hole #1



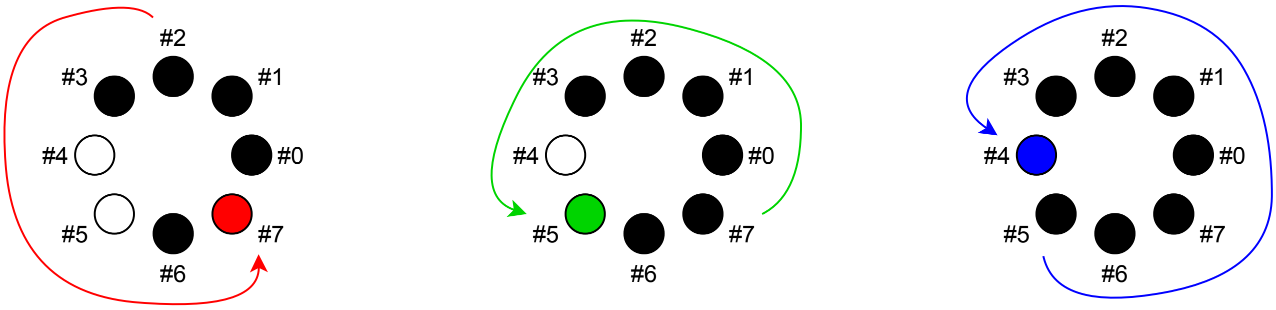

Move $i = 2$ holes forward, and fill the destination hole #3




Move $i = 3$ holes forward, and fill the destination hole #6




Move $i = 4$ holes forward, and fill the destination hole #2



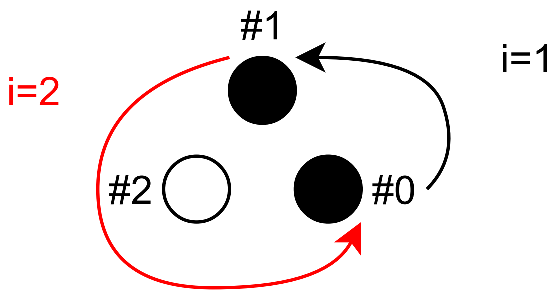

Last steps: We repeat the process, moving forward:

- $i = 5$ holes (to #7),
- then $i = 6$ holes (to #5),
- and finally $i = 7$ holes (to #4).

At this point all holes have been filled, so we stop. The order in which we filled the holes: [ #0, #1, #3, #6, #2, #7, #5, #4 ] can be seen as a permutation of the first $N = 8$ non-negative integers.

## Success and failure

Repetitions are not accepted, i.e., whenever we land on a hole that has already been filled, we call that a failure. Example: $N = 3$:




Failure for $N = 3$

Looking at the first few values of $N$:

| | | |
|---|---|---|
| N=2 | success | |
| N=3 | | failure |
| N=4 | success | |
| N=5 | | failure |
| N=6 | | failure |
| N=7 | | failure |
| N=8 | success | |
| N=9 | | failure |
| ... | | failure |
| N=15 | | failure |
| N=16 | success | |
| N=17 | | failure |

Successes seem to correspond to powers of two: $N = 2^q$ where $q \in \mathbb{N}_+^*$

# Main result

**A circular set of $N$ elements can be visited in the above–described non–uniform way if and only if $N$ is a power of two ($N = 2^q$).**

Formally: we define the property:

> $P_N \triangleq$ "for the circular set of $N$ holes, for each step $i = 1 \ldots N$, we land on an empty hole (an thus after the $N$ steps all holes are filled)".

The main result of the present paper is:

$$N\ is\ a\ power\ of\ two\ \Leftrightarrow\ P_N\ true$$

where "$N$ is a power of two" means $log_2(N) \in \mathbb{N}_+$ or equivalently: $\exists\, q \in \mathbb{N}_+ \; s.t. \; N = 2^q$.

Appendix (A1) demonstrates that $N \; power \; of \; 2 \Rightarrow \; P_N \; true$.

Appendix (A2) demonstrates that $N \; not \; a \; power \; of \; 2 \Rightarrow \; P_N \; false$.

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# Openings

## (O1): Filling, seen as a permutation

The order in which we fill the holes, e.g. for $N = 8$: [ #0, #1, #3, #6, #2, #7, #5, #4 ] can be seen as a permutation of the first $N = 8$ non-negative integers.

What happens if we repeat this permutation?

```
N=8 (2^3)

step  0  current  0,1,2,3,4,5,6,7
step  1  current  0,1,3,6,2,7,5,4
step  2  current  0,1,6,5,3,4,7,2
step  3  current  0,1,5,7,6,2,4,3
step  4  current  0,1,7,4,5,3,2,6
step  5  current  0,1,4,2,7,6,3,5
step  6  current  0,1,2,3,4,5,6,7

=> period: 6
```

So we can observe a periodicity. What about other values of $N = 2^q$?

```
N=   4 (2^2)     period:      2
N=   8 (2^3)     period:      6
N=  16 (2^4)     period:     14
N=  32 (2^5)     period:     30
N=  64 (2^6)     period:   2280
N= 128 (2^7)     period:  18480
```

```
N= 256 (2^8)     period:    2964
N= 512 (2^9)     period:   10248
N=1024 (2^10)    period: 6036022
```

These results were obtained using this JavaScript code, running it in a browser console.

*Open questions:* Do all power of two $N = 2^q$ have a periodicity? If yes, can someone derive a formula giving the period as a function of $N$, i.e. the series (2,6,14,30,2280,...)?

# (O2): About other filling methods

This paper investigated the particular filling method, where at each step $i$ we move $j = i$ holes forward, thus defining the series:

$$(j)_i = (1, 2, 3, \ldots, N-1)$$

Besides the obvious "uniform" filling method, where we move 1 hole forward each time:

$$(j)_i = (1, 1, 1, \ldots, 1)$$

...are there other "non-uniform" filling methods without repetition, at least for $N$ being a power of two?

For example, for $N = 2^2 = 4$ the answer is yes. Besides the filling method investigated so far:

$$(j)_i = (1, 2, 3)$$

there is also:

$$(j)_i = (3, 2, 1)$$

which is equivalent to invert the direction. If we additionally restrict $(j)_i$ being itself a permutation of $(1, 2, 3, \ldots, N-1)$, these are the only two possibilities for $(j)_i$ for $N = 4$.

*Open question:* Are there other methods $(j)_i$ to fill without repetition, which work for all $N$ powers of two? Especially when we restrict the series $(j)_i$ being itself a permutation of $(i)_i = (1, 2, 3, \ldots, N-1)$ ?

# Appendices

## (A1) Show that $N\ power\ of\ 2 \Rightarrow\ P_N\ true$

Let us assume $H_1$ and $H_2$:

$H_1$
: $N$ is a power of two: $\exists\ q\ \in \mathbb{N}^*_+\ s.t.\ N = 2^q$

$H_2$
: $P_N\ false$, i.e. in at least one of the $N$ steps $i = 1 \ldots N$, we land on a hole that has already been filled:

$$\exists (a,b) \in \mathbb{N}^2\ s.t.\ 0 \leq a < b < N\ and\ V_a \equiv V_b\ [N]$$

where:

- $V_i$ is, at step $i$, the total number of holes we've been moving since the beginning:

$$V_i \triangleq \sum_{j=1}^{i} j = \frac{i(i+1)}{2}$$

- $V_a \equiv V_b\ [N]$ means congruence modulo $N$:

$$\exists k \in \mathbb{N}\ s.t.\ V_b - V_a = k \cdot N$$

$H_2$ implies:

$$\exists k \in \mathbb{N}\ s.t.\ \frac{b(b+1)}{2} - \frac{a(a+1)}{2} = k \cdot N$$

which we can rewrite:

$$b^2 - a^2 + b - a = 2 \cdot k \cdot N$$
$$(b-a) \cdot (b+a+1) = 2 \cdot k \cdot N \qquad (\alpha)$$

Observations: $b-a$ and $b+a$ have same parities, hence $b-a$ and $b+a+1$ have opposite parities, i.e. one is odd and the other one is even.

On the right hand side, $2 \cdot N = 2^{q+1}$ is a power of two, thus $k$ must be odd. Moreover, since $(b-a)$, $(b+a+1)$ and $2 \cdot N$ are all non-negative, we have $k > 0$.

Summarized: $k \geq 1$, and $k$ is the only odd term on the right hand side.

Let us assume $k = 1$. Because the two terms on the left hand side have opposite parities, then either $a + b + 1 = 1$ (impossible), or $b - a = 1$ i.e. $b = a + 1$ so $(\alpha)$ can be written: $1 \cdot 2 \cdot b = 1 \cdot 2 \cdot N$, and thus $b = N$, which is impossible as well.

Therefore: $k$ is odd and $k \geq 3$.

## (A1.a) Let us assume $b - a = k$

$(\alpha)$ can be rewritten:

$$k \cdot (1 + 2a + k) = 2 \cdot k \cdot N$$

$$1 + 2a + k = 2 \cdot N$$

$$a = N - \frac{k + 1}{2}$$

$$b = a + k = N + \frac{2k - k - 1}{2}$$

$$b = N + \frac{k - 1}{2}$$

Since $k \geq 3$, we have $b > N$, which is impossible.

## (A1.b) Let us assume $a + b + 1 = k$

i.e.

$$b - a = b - (k - (b + 1)) = 2b - k + 1$$

$(\alpha)$ can be rewritten:

$$(2b - k + 1) \cdot k = 2 \cdot k \cdot N$$

$$2b - k + 1 = 2N$$

$$b = N + \frac{k - 1}{2}$$

Since $k \geq 3$, we have $b > N$, which is impossible.

## Conclusion of (A1)

For $q \in \mathbb{N}_+^*$ and $N = 2^q$, assuming $P_N$ false leads to a contradiction, therefore $P_N$ is true. $P_{2^0} = P_1$ is obvious, therefore:

$$\forall q \in \mathbb{N}_+ \; P_{2^q} \; true$$

## (A2) Show that $N\ not\ a\ power\ of\ 2 \Rightarrow\ P_N\ false$

Formally: we want to show that:

$$\forall N \in \mathbb{N}_+^*\ s.t.\ log_2 N \notin \mathbb{N}$$
$$\exists (a,b) \in \mathbb{N}^2 \quad 0 \le a < b < N \quad s.t. \quad V_a \equiv V_b\ [N]$$

where $V_i \triangleq \frac{i(i+1)}{2}$ is the number of holes we've moved since the beginning.

For $(a,b) \in \mathbb{N}^2$, we define the property:

$$T_{a,b} \quad \triangleq \quad 0 \le a < b < N \ \ and \ \ V_a \equiv V_b\ [N]$$

To prove the result, we need to find at least one value of $(a,b)$ that verifies $T_{a,b}$

### (A2.1) Case: $N = 2p+1$ where $p \in \mathbb{N}_+^*$

Since $V_i \triangleq \frac{i(i+1)}{2}$ we can write:

$$V_{p+1} - V_{p-1} = p+1\ +\ p = N$$

Thus, $a = p-1$ and $b = p+1$ verify $T_{a,b}$.

### Transition

It remains to find $(a,b)$ verifying $T_{a,b}$ when $N = 2p$ is not a power of two.

### (A2.2) Case: $N$ even but not a power of two

i.e.

$$\exists (p,q) \in (\mathbb{N}_+^*)^2 \qquad N = 2^q \cdot (2p+1)$$

e.g.

```
N =  6 = 2*3   q:1   p:1
N = 10 = 2*5   q:1   p:2
N = 12 = 4*3   q:2   p:1
```

```
N = 14 = 2*7   q:1   p:3
```

**(A2.2.1) When $p \geq 2^q$**

Let $a = p - 2^q$ and $b = p + 2^q$.

We have $0 \leq a$ and $a < b$ and $N - b = 2^q \cdot 2p - p = p(2^{q+1} - 1) > 0$

therefore we have $0 \leq a < b < N$.

Besides,

$$V_b - V_a = \sum_{j=1}^{p+2^q} j \quad - \quad \sum_{j=1}^{p-2^q} j$$

$$= \sum_{j=p-2^q}^{p+2^q} j \quad - \quad (p - 2^q)$$

$$= p \cdot (2 \cdot 2^q + 1) - (p - 2^q)$$

$$= 2^q \cdot (2p + 1)$$

$$= N$$

$(a, b)$ verify $T_{a,b}$.

**(A2.2.2) When $p < 2^q$**

Let $a = 2^q - p - 1$ and $b = 2^q + p$.

We can show, as in (A2.2.1), that $0 \leq a < b < N$.

Besides,

$$V_b - V_a = \sum_{j=1}^{2^q+p} j \quad - \quad \sum_{j=1}^{2^q-p-1} j$$

$$= \sum_{j=2^q-p}^{2^q+p} j = 2^q \cdot (2p + 1)$$

$$= N$$

Thus $(a, b)$ verify $T_{a,b}$.